



Short Note

High order ADI method for solving unsteady convection–diffusion problems

Samir Karaa¹, Jun Zhang^{*}

Laboratory for High-Performance Scientific Computing and Computer Simulation, Department of Computer Science, University of Kentucky, 773 Anderson Hall, Lexington, KY 40506-0046, USA

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Abstract

We propose a high order alternating direction implicit (ADI) solution method for solving unsteady convection–diffusion problems. The method is fourth order in space and second order in time. It permits multiple use of the one-dimensional tridiagonal algorithm with a considerable saving in computing time, and produces a very efficient solver. It is shown through a discrete Fourier analysis that the method is unconditionally stable for 2D problems. Numerical experiments are conducted to test its high accuracy and to compare it with the standard second-order Peaceman–Rachford ADI method and the spatial third-order compact scheme of Noye and Tan.

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1. Introduction

We consider the unsteady two-dimensional (2D) convection–diffusion equation for a transport variable u

$$\frac{\partial u}{\partial t} - \alpha_x \frac{\partial^2 u}{\partial x^2} - \alpha_y \frac{\partial^2 u}{\partial y^2} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} = 0, \quad \text{in } \Omega \times (0, T], \quad (1a)$$

$$u(x, y, t) = g(x, y, t), \quad (x, y) \in \partial\Omega, \quad t \in (0, T], \quad (1b)$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \quad (1c)$$

^{*} Corresponding author. Tel.: +1-859-2573892; fax: +1-859-3231971.

E-mail addresses: samir@csr.uky.edu, skaraa@squ.edu.om (S. Karaa), jzhang@cs.uky.edu (J. Zhang).

URL: <http://www.cs.uky.edu/~jzhang>.

¹ Present address: Department of Mathematics and Statistics, Sultan Qaboos University, PO Box 36, Al-Khod 123, Muscat, Sultanate of Oman.

where $\Omega \subset \mathfrak{R}^2$ is a rectangular domain, $(0, T]$ is the time interval, and g and u_0 are given functions of sufficiently smoothness. In (1a), c_x and c_y are constant speeds of convection and $\alpha_x > 0$ and $\alpha_y > 0$ are constant diffusivities in the x - and y -direction, respectively. This equation may be seen in computational hydraulics and fluid dynamics to model convection–diffusion of quantities such as mass, heat, energy, vorticity, etc [10].

Various numerical finite difference schemes have been proposed to solve convection–diffusion problems approximately. Most of these schemes are either first-order or second-order accurate in space, and have poor quality for convection dominated flows if the mesh is not sufficiently refined. Higher order discretizations are generally associated with large (non-compact) stencils which increase the band-width of the resulting matrix and lead to a large number of arithmetic operations, especially for higher dimensional problems.

To obtain satisfactory higher order numerical results with reasonable computational cost, there have been attempts to develop higher order compact (HOC) schemes, which utilize only the grid nodes directly adjacent to the central node. After deriving several higher order implicit schemes for unsteady one-dimensional convection–diffusion equations [5], Noye and Tan [6] proposed a nine-point HOC implicit scheme for unsteady 2D convection–diffusion equations with constant coefficients. The scheme is third-order accurate in space and second-order accurate in time, and has a large zone of stability. Two other classes of compact difference schemes of order 2 in time and order 4 in space have been derived in [8,9], with different choices of weighting parameters.

The 2D HOC scheme proposed in [2] for solving steady state equations, and analyzed for instance in [4,12,13], was extended by Spitz and Carey [11] to solve unsteady 1D convection–diffusion equations with variable coefficients and 2D diffusion equations. Recently, based on the work of [11], a class of HOC schemes with weighted time discretization have been derived for solving unsteady 2D convection–diffusion equations [3]. The numerical experiments presented in [3] show that these schemes are accurate and capture very well the transient solutions of convection–diffusion problems.

In this paper, we propose a high order alternating direction implicit solution method for solving unsteady 2D convection–diffusion problems. We make the alternating direction implicit (ADI) splitting possible, by choosing a spatial discretization different from the one derived in [2], and used in [3,11] for unsteady problems. The new ADI method is second order in time and fourth order in space and does not need much more work than the well-known Peaceman–Rachford (P–R) ADI method [7].

2. Derivation of compact ADI scheme

We start by examining the one-dimensional steady convection diffusion equation

$$-\alpha_x \frac{d^2 u}{dx^2} + c_x \frac{du}{dx} = f, \quad (2)$$

where c_x and $\alpha_x > 0$ are constants and f is a function of x . Using the techniques outlined in [12,14,15], it is easy to derive a three-point fourth-order compact scheme for Eq. (2) as

$$-\left(\alpha_x + \frac{c_x^2 \Delta x^2}{12\alpha_x}\right) \delta_x^2 u_i + c_x \delta_x u_i = \left(1 + \frac{\Delta x^2}{12} \left(\delta_x^2 - \frac{c_x}{\alpha_x} \delta_x\right)\right) f_i + O(\Delta x^4), \quad (3)$$

where δ_x^2 and δ_x are the first- and second-order central difference operators, Δx is the mesh size.

For convenience, we define two finite difference operators

$$L_x = 1 + \frac{\Delta x^2}{12} \left(\delta_x^2 - \frac{c_x}{\alpha_x} \delta_x\right), \quad A_x = -\left(\alpha_x + \frac{c_x^2 \Delta x^2}{12\alpha_x}\right) \delta_x^2 + c_x \delta_x.$$

Eq. (3) can then be formulated symbolically as

$$L_x^{-1}A_x u_i = f_i + O(\Delta x^4).$$

This symbolic construction can be used to derive high order compact schemes for higher dimensional problems [14]. When applied to the 2D convection diffusion equation

$$-\alpha_x \frac{\partial^2 u}{\partial x^2} - \alpha_y \frac{\partial^2 u}{\partial y^2} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} = f, \quad (4)$$

with $c_x, c_y, \alpha_x > 0$ and $\alpha_y > 0$ being constant and f a smooth function of x and y , it yields the following fourth-order approximation

$$\left(L_x^{-1}A_x + L_y^{-1}A_y \right) u_{ij} = f_{ij} + O(\Delta^4), \quad (5)$$

where $O(\Delta^4)$ denotes the $O(\Delta x^4) + O(\Delta y^4)$ term. Here, the meaning of the notations L_y and A_y is obvious. We assume that the equation is approximated on a uniform grid with mesh sizes Δx and Δy in the x - and y -direction, respectively. Applying to both sides of Eq. (5) with the operator $L_x L_y$, we obtain

$$(L_y A_x + L_x A_y) u_{ij} = L_x L_y f_{ij} + O(\Delta^4). \quad (6)$$

This approximation is clearly fourth-order accurate, and has a compact nine-point stencil since it involves only discrete difference operators of the form $\delta_x^p \delta_y^q$, where p and q are non-negative integers less than or equal to 2. Notice that in deriving (6) we used the fact that the two operators L_x and L_y commute with each other, which is possible since the convection and diffusion terms are assumed constant.

A fourth-order semi-discrete approximation to the unsteady convection–diffusion equation in (1) can be obtained by replacing f with $-\partial u / \partial t$ in (6)

$$L_x L_y \frac{\partial u^n}{\partial t} = -(L_y A_x + L_x A_y) u^n + O(\Delta^4),$$

where u^n is the approximate solution at time $t^n = n\Delta t$, $n \geq 0$ and Δt is the time increment. Employing Crank-Nicolson time discretization, we have

$$L_x L_y \frac{u^{n+1} - u^n}{\Delta t} = -\frac{1}{2} (L_y A_x + L_x A_y) (u^{n+1} + u^n) + O(\Delta^4) + O(\Delta t^2). \quad (7)$$

This discretization is obviously of order 2 in time and order 4 in space, due to the use of the Crank-Nicolson type integrator in time with a fourth-order spatial discretization. After rearrangement and multiplying (7) by Δt , we have

$$\left(L_x L_y + \frac{\Delta t}{2} (L_y A_x + L_x A_y) \right) u^{n+1} = \left(L_x L_y - \frac{\Delta t}{2} (L_y A_x + L_x A_y) \right) u^n + O(\Delta t \Delta^4) + O(\Delta t^3). \quad (8)$$

To get a solution for our problem, we must solve at each time step a sparse linear system arising from the implicit discretization (8). Direct methods based on Gaussian elimination may be too expensive to use for solving such sparse linear system of large size. Iterative methods, such as Krylov subspace methods, are generally efficient, however, they may be expensive to use at each time step, and especially for higher dimensional problems.

A way around in developing an efficient solution method to our problem is to solve a perturbed problem which has the same order of accuracy as (8) and which allows to reduce the 2D problem to a succession of many one-dimensional problems. To accomplish this, we add the terms $\Delta t^2 A_y A_x u^{n+1} / 4$ and $\Delta t^2 A_x A_y u^n / 4$ to the left- and right-hand sides of (8), respectively, so that Eq. (8) becomes after dropping the error terms

$$\left(L_x L_y + \frac{\Delta t}{2}(L_y A_x + L_x A_y) + \frac{\Delta t^2}{4} A_y A_x\right) u^{n+1} = \left(L_x L_y - \frac{\Delta t}{2}(L_y A_x + L_x A_y) + \frac{\Delta t^2}{4} A_x A_y\right) u^n,$$

which can be factored as

$$\left(L_x + \frac{\Delta t}{2} A_x\right) \left(L_y + \frac{\Delta t}{2} A_y\right) u^{n+1} = \left(L_x - \frac{\Delta t}{2} A_x\right) \left(L_y - \frac{\Delta t}{2} A_y\right) u^n. \quad (9)$$

The extra term added to Eq. (8) is given by

$$\begin{aligned} \frac{\Delta t^2}{4} A_y A_x (u^{n+1} - u^n) &= \frac{\Delta t^3}{4} A_y A_x \frac{\partial u^n}{\partial t} + \mathcal{O}(\Delta t^4) \\ &= \frac{\Delta t^3}{4} \left(c_y - \alpha_y \frac{\partial}{\partial y}\right) \left(c_x - \alpha_x \frac{\partial}{\partial x}\right) \frac{\partial^3 u^n}{\partial t \partial x \partial y} + \mathcal{O}(\Delta t^3 \Delta^2) + \mathcal{O}(\Delta t^4). \end{aligned} \quad (10)$$

If $\Delta t \leq \min(\Delta x, \Delta y)$, we added to Eq. (8) a term which is of similar order to its truncation error. It then follows that the resulting approximation (9) is second order in time and fourth order in space. Introducing an intermediate variable u^* and applying the D'Yakonov ADI-like scheme [1], Eq. (9) leads to

$$\left(L_x + \frac{\Delta t}{2} A_x\right) u^* = \left(L_x - \frac{\Delta t}{2} A_x\right) \left(L_y - \frac{\Delta t}{2} A_y\right) u^n, \quad (10a)$$

$$\left(L_y + \frac{\Delta t}{2} A_y\right) u^{n+1} = u^*. \quad (10b)$$

We note that the intermediate values of u^* at the boundary are obtained using Eq. (10b).

To study the stability of the new scheme, we use the von Neumann linear stability analysis. If we let $u_{ij}^n = b^n e^{i\theta_x i} e^{j\theta_y j}$ to be the value of u^n at node (i, j) , where $\mathbf{I} = \sqrt{-1}$, b^n is the amplitude at time level n , and $\theta_x (= 2\pi\Delta x/A_1)$ and $\theta_y (= 2\pi\Delta y/A_2)$ are phase angles with wavelengths A_1 and A_2 , respectively, the amplification factor $\xi(\theta_x, \theta_y) = b^{n+1}/b^n$, for stability, has to satisfy the relation $|\xi(\theta_x, \theta_y)| \leq 1$, for all θ_x and θ_y in $[-\pi, \pi]$. By substituting the expressions of u_{ij}^n and u_{ij}^{n+1} in (9), the amplification factor is found to be

$$\xi(\theta_x, \theta_y) = g_x(\theta_x) g_y(\theta_y),$$

where $g_x(\theta_x)$ is given by

$$g_x(\theta_x) = \frac{(\gamma_1 - \gamma_2) + \mathbf{I}(\gamma_3 + \gamma_4)}{(\gamma_1 + \gamma_2) + \mathbf{I}(\gamma_3 - \gamma_4)}$$

with

$$\gamma_1 = 1 - \frac{1}{3} \sin^2 \frac{\theta_x}{2}, \quad \gamma_2 = 2\Delta t \left(\frac{1}{\Delta x^2} + \frac{c_x^2}{12} \right) \sin^2 \frac{\theta_x}{2}, \quad \gamma_3 = \frac{c_x \Delta t}{12} \sin \theta_x, \quad \gamma_4 = \frac{c_x \Delta t}{2\Delta x} \sin \theta_x,$$

being all non-negative. The other term $g_y(\theta_y)$ is defined in a similar way by replacing x by y in the above expressions. It is easy to verify that $|g_x(\theta_x)| \leq 1$ for all $\theta_x \in [-\pi, \pi]$ if and only if $\gamma_1 \gamma_2 \geq \gamma_3 \gamma_4$. A simple calculation shows that

$$\gamma_1 \gamma_2 = 2\Delta \left(1 - \frac{1}{3} \sin^2 \frac{\theta_x}{2} \right) \left(\frac{1}{\Delta x^2} + \frac{c_x^2}{12} \right) \sin^2 \frac{\theta_x}{2} \geq c_x^2 \frac{\Delta t}{6} \left(1 - \frac{1}{3} \sin^2 \frac{\theta_x}{2} \right) \sin^2 \frac{\theta_x}{2},$$

and

$$\gamma_3\gamma_4 = c_x^2 \frac{\Delta t}{6} \left(1 - \sin^2 \frac{\theta_x}{2}\right) \sin^2 \frac{\theta_x}{2}.$$

Hence, $\gamma_3\gamma_4 \leq \gamma_1\gamma_2$ and it follows that $|g_x(\theta_x)| \leq 1$ for all $\theta_x \in [-\pi, \pi]$. Since we have a similar inequality for $g_y(\theta_y)$, we conclude that the new method is unconditionally stable.

3. Numerical experiments

We first examine a diffusion problem in the unit square domain $[0, 1] \times [0, 1]$, with diffusion coefficients $\alpha_x = \alpha_y = 1$ (and $c_x = c_y = 0$). The exact solution of this test problem is given by

$$u(x, y, t) = e^{-2\pi^2 t} \sin(\pi x) \sin(\pi y).$$

The initial and Dirichlet boundary conditions are directly taken from this solution. We consider uniform grids with different mesh sizes and compare the accuracy of the computed solutions from the present ADI scheme and the P–R ADI scheme. The quantity that we compare is the L^2 -norm error of the computed solution with respect to the exact solution. We choose a time step $\Delta t = 0.001$ and $t = 1$ for the entire simulation process.

In Fig. 1, we plot the L^2 -norm errors at each time step in each case. The figure shows the superiority of the present ADI scheme over the Peaceman–Rachford ADI scheme. The error obtained on a 10×10 grid is much smaller than the one obtained using the Peaceman–Rachford ADI scheme on a 40×40 grid.

To further study the validity and effectiveness of the new high order ADI method, we apply (10) to a special problem defined in the square region $[0, 2] \times [0, 2]$, with an analytical solution given, as in [6], by

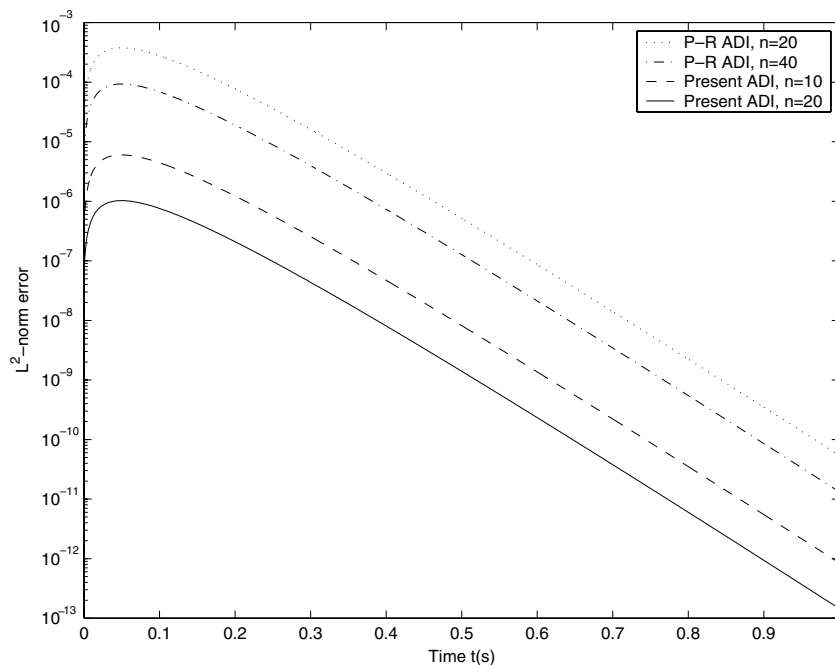


Fig. 1. Comparison of the L^2 -norm errors produced by the present scheme and the P–R scheme at each time step.

Table 1

L^2 -norm errors at $t = 1.25$ s and CPU times delivered by four different schemes, with $\Delta t = 0.00625$ and $\Delta x = \Delta y = 0.025$

Method	L^2 -norm error	CPU time (s)
Noye and Tan	1.24×10^{-4}	26.4
P–R ADI	2.02×10^{-3}	3.4
Kalita et al.	1.02×10^{-4}	19.7
Present ADI	5.62×10^{-5}	3.5

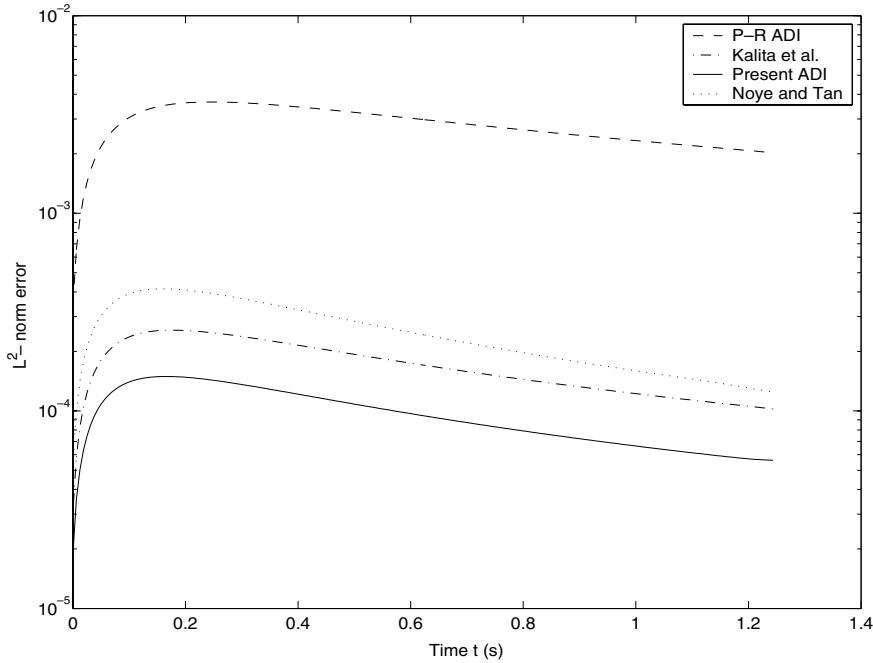


Fig. 2. Comparison of the L^2 -norm errors produced by four different schemes at each time step.

$$u(x, y, t) = \frac{1}{4t + 1} \exp \left[-\frac{(x - c_x t - 0.5)^2}{\alpha_x(4t + 1)} - \frac{(y - c_y t - 0.5)^2}{\alpha_y(4t + 1)} \right].$$

The Dirichlet boundary and the initial conditions are directly taken from this solution.

The L^2 -norm of the errors produced by the present scheme, the P–R ADI scheme [7], the spatial third-order nine-point compact scheme of Noye and Tan [6], and the fourth-order nine-point compact scheme of Kalita et al. [3],² are presented in Table 1, with the total elapsed time (CPU) in seconds delivered in each case. The results show that the present ADI scheme provides the most accurate solution. In Fig. 2, we plot the L^2 -norm errors at each time step for the entire simulation process in each case. The figure shows that the four errors have the same behavior with the error of the present scheme remaining smaller than the other errors at every time step. Contour plots of the exact and numerically approximated pulses in the sub-region $1 \leq x, y \leq 2$ are drawn in Fig. 3 for each test carried out (except the Kalita et al. [3] scheme, for which the

² The computed data in [3] are incorrect, which has been confirmed with the lead author of [3].

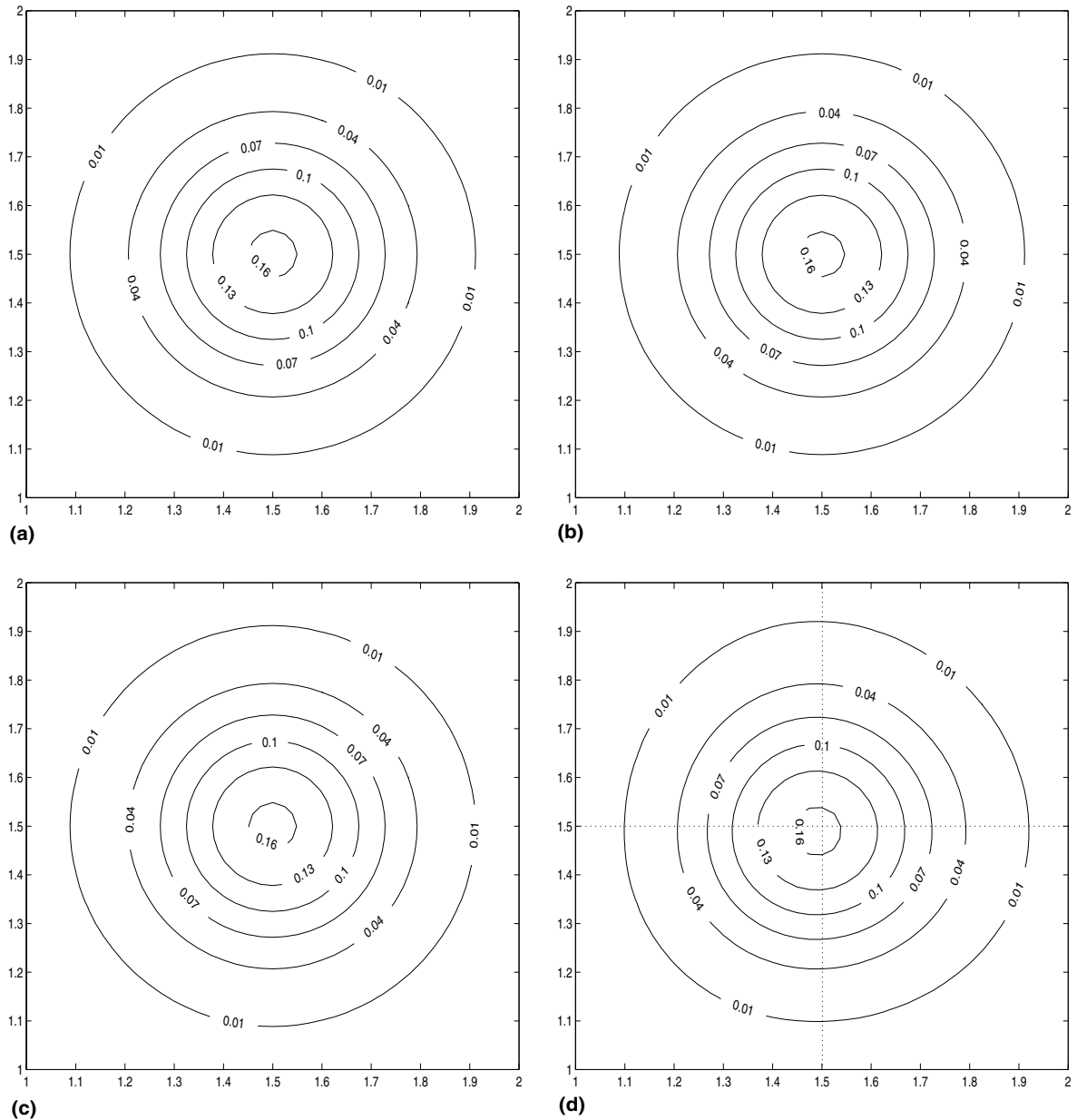


Fig. 3. Contour plots of the pulse in the sub-region $1 \leq x, y \leq 2$ at $t = 1.25$ s: (a) exact, (b) Noye and Tan scheme, (c) present ADI scheme, and (d) P-R ADI scheme, with $\Delta t = 0.00625$.

contour plot can be found). Figs. 3(b) and (c) show that the present scheme as well as the Noye and Tan scheme capture very well the moving pulse, yielding pulses centered at $(1.5, 1.5)$ and almost indistinguishable from the exact one displayed in Fig. 3(a). The Peaceman–Rachford ADI scheme produces, however, a pulse distorted in the x - and y -directions, owing to the fact the second-order error terms of the method is related to the wave numbers in both directions, as is explained in [6].

Table 1 shows that the two ADI schemes deliver very small CPU times. Almost eight times smaller than the one delivered by the Noye and Tan scheme. Hence, the new scheme is the most effective in term of accuracy and time consumption. We notice that, the ADI methods are carried out by repeatedly solving a series of triangular linear systems, while to solve the linear system arising from the Noye and Tan discretization (and the discretization of Kalita et al.), we used a preconditioned iterative solver (GMRES with ILU(0)). The iterations are terminated when the 2-norm of the relative residual is reduced by a factor of 10^7 . All computer programs are written in standard Fortran 77 and were run on a SunBlade 100 machine.

4. Concluding remarks

We proposed a high order accurate alternating direction implicit solution method for solving unsteady convection–diffusion problems. The method is fourth order in space and second order in time and allows a considerable saving in computing time. It is shown through a discrete Fourier analysis that it is unconditionally stable for 2D problems. The method is easily extendible to multi-dimensional problems. Numerical experiments are conducted to test its high accuracy and to show its superiority over the classical Peaceman–Rachford ADI scheme and the spatial third-order compact scheme of Noye and Tan, in terms of accuracy and computational cost.

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